

Fish Population Dynamics with Harvesting and Toxicant Effects

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USM

- ▶ Population modelling is a mathematical approach to study population dynamics.
- ▶ Population dynamics describe how a population fluctuates in size and composition over time, in order to forecast future changes.
- ▶ These dynamics are useful for ecologists/ biologists/ mathematicians to determine the threshold level at which specific/general populations can be conserved.
- ▶ Some models consider growth without environmental constraints while others deliberate the surrounding resources.

- ▶ In population models, interactions between population and its environment, individual and other species should be taken into account.



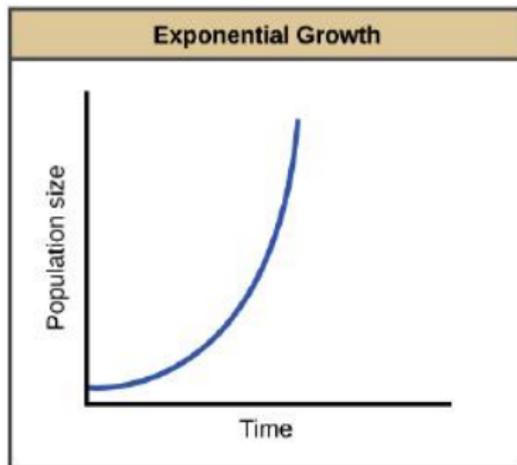
(a) Prey-predation



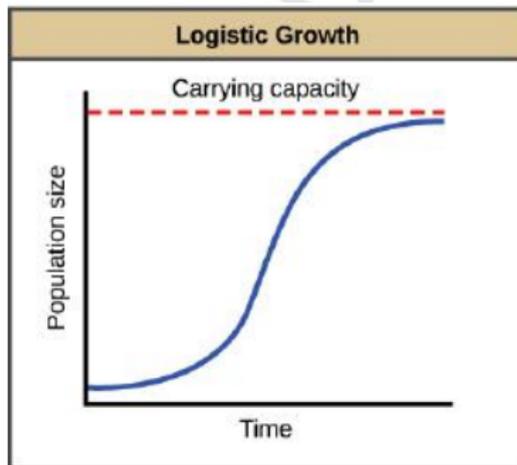
(b) Competition

Figure 1 : Interactions among fish populations

- ▶ Population growth rate can be defined as the rate of change of population size over time
- ▶ Exponential growth rate: $\frac{dN}{dt} = rN$ represents constant population growth rate regardless of the population density.
- ▶ Logistic growth: $\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)$ describes the population growth rate decreases as the population density approached maximum imposed by limited resources or environmental carrying capacity K .



(a) Exponential Growth



(b) Logistic Growth

- ▶ Stability analysis is performed to address the stability of the equilibria that exist in a dynamical system under small perturbation of initial conditions.
- ▶ Local stability refers to the solutions that tend to approach the equilibrium point under initial conditions close to the equilibrium points.
- ▶ Local stability analysis of the equilibria can be examined by constructing the Jacobian Matrix and the corresponding eigenvalue.
- ▶ Global stability refers to the solutions that must approach the equilibrium point regardless of the initial conditions.
- ▶ Global stability of a dynamical system can be studied by constructing the Lyapunov function.

- ▶ Bifurcation can occur when a slight fluctuation in the parameter causes significant change in its stability behaviors.
- ▶ Bifurcation can happen in ODEs, DDEs and PDEs.
- ▶ Bifurcation analysis uses the technique of parameter variation to examine the topological change in the dynamical behaviors.
- ▶ Singularity analysis or sensitivity analysis are some approaches used in investigating bifurcation of a system.
- ▶ Bifurcation analysis needs analytical solutions and numerical simulations to understand the system's behaviour.
- ▶ A parameter that gives significant change in the system is called the main primary bifurcation parameter, and all other parameters, which are often called secondary bifurcation parameters.

- ▶ The concept of **carrying capacity** is extensively applied in many areas of study.
- ▶ Carrying capacity is the maximum population abundance (for a given species) an environment can sustain.
- ▶ The carrying capacity, K is usually regarded as a constant in population growth models which is not often realistic.

A non-autonomous logistic equation is the initial model to describe changing environment where the carrying capacity is **time-dependent**:

$$N' = rN \left(1 - \frac{N}{K(t)} \right),$$

Periodic: $K(t) : a + b \sin(ct + \psi)$

Saturation: $K(t) : a + b(1 - e^{-ct})$

Logistic: $K(t) : K_1 + K_2/(1 + ae^{-bt})$

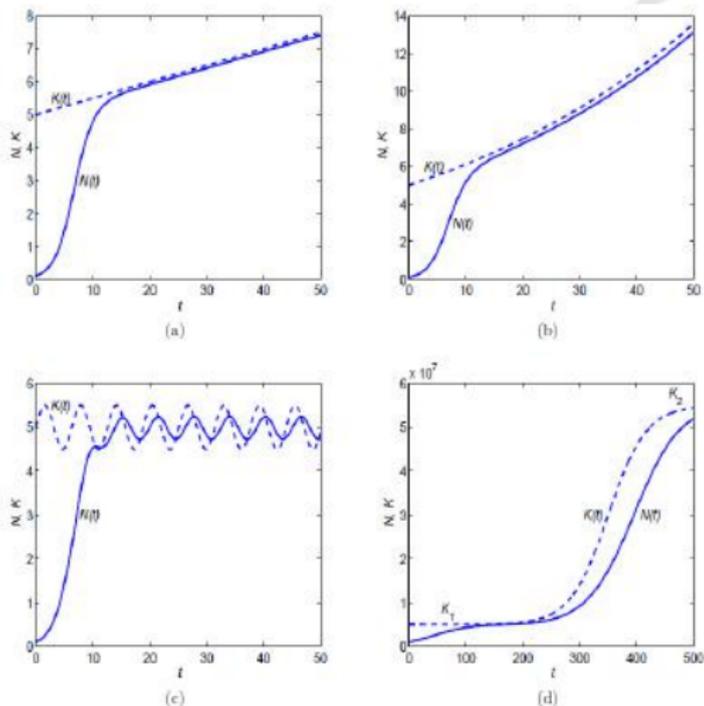


Figure 2 : Logistic model with time-dependent carrying capacities.

The modified logistic model with the carrying capacity as **state-variable**:

$$N' = aN \left(1 - \frac{N}{K} \right),$$

Decay: $K' = -bN.$

Open-ended: $K' = b(N - K).$

Interaction: $K' = bK - cKN.$

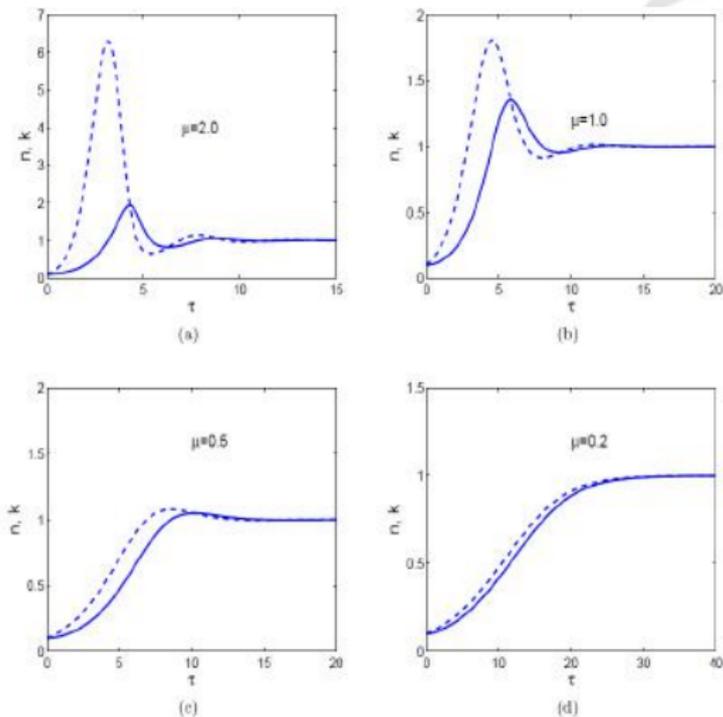


Figure 3 : Logistic model with state variable carrying capacities.

- ▶ Single population models provide the dynamical behavior of the inter-relation between the population and its carrying capacity.
- ▶ Two species model will give richer nonlinear dynamics.
- ▶ In the context of **prey (X)** and **predator (Y)** model, sharing the same **base resource (Z)** is one of important cases to consider.

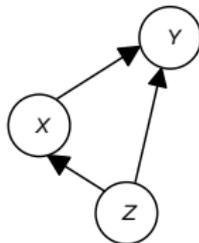


Figure 4 : Basic food web system.

Examples of predator-prey-resource system⁴:

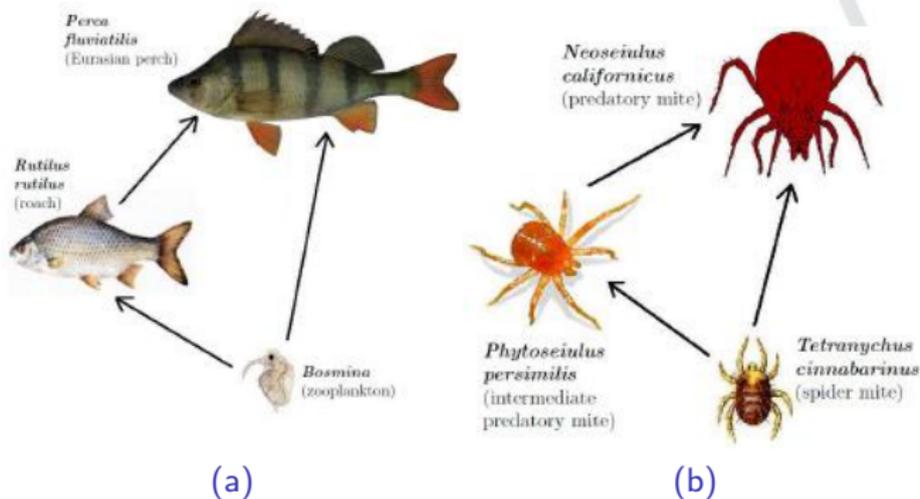


Figure 5 : (a) Fish populations, (b) Spider mite populations.

The non-dimensionalised version

$$\begin{aligned}x' &= x \left(1 - \frac{x}{k}\right) - xy, \\y' &= \alpha y \left(1 - \frac{\beta y}{k}\right) + xy, \\k' &= k (\gamma - \delta x - \epsilon y).\end{aligned}\tag{1}$$

x : Prey population

y : Predator population

k : Resource

Environmental carrying capacity as biotic resource enrichment.

There are 3 equilibria P_i in the (x, y, k) phase space:

Steady states	Characteristics	Stability
$P_1(\frac{\gamma}{\delta}, 0, \frac{\gamma}{\delta})$	Extinction of predator	Unstable
$P_2(0, \frac{\gamma}{\epsilon}, \frac{\beta\gamma}{\epsilon})$	Extinction of prey	Stable if $\gamma > \epsilon$
$P_3(x^*, y^*, k^*)$	Coexistence	Stable if $\gamma < \epsilon$

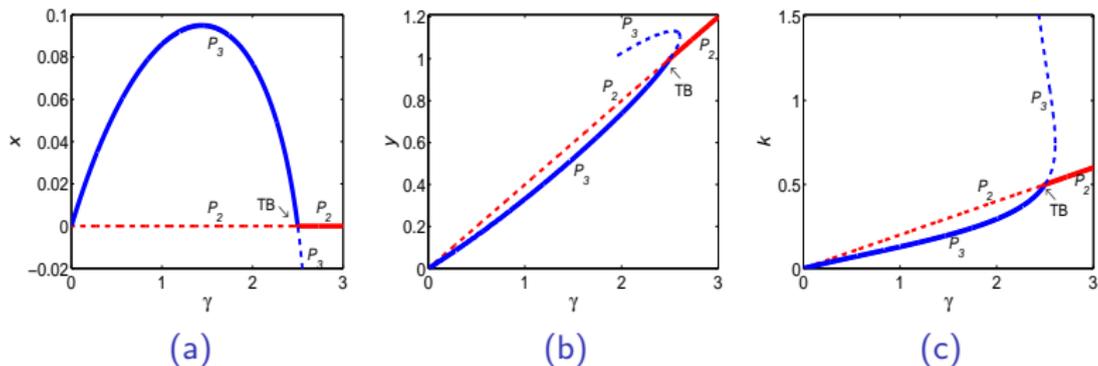


Figure 6 : Steady state diagrams for (a) x ; (b) y ; (c) k with $\alpha = 0.3$.

TB : Transcritical Bifurcation;

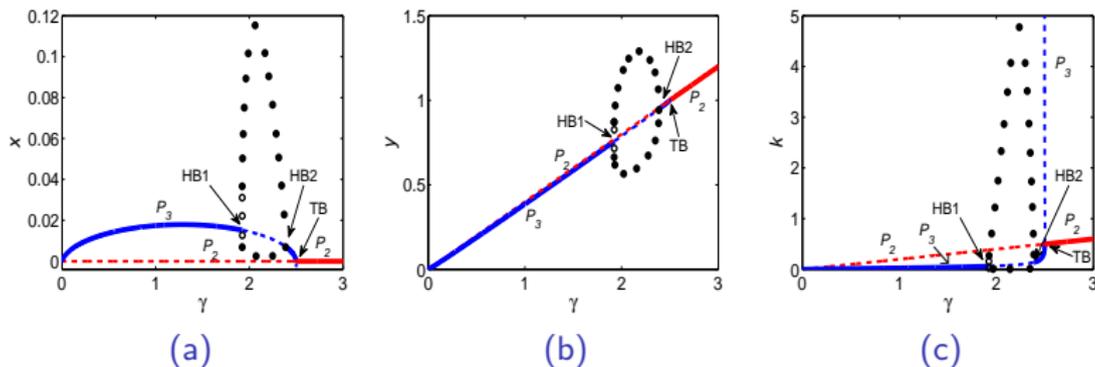


Figure 7 : Steady state diagrams for (a) x ; (b) y ; (c) k with $\alpha = 0.003$.

TB : Transcritical Bifurcation; **HB** : Hopf Bifurcation;

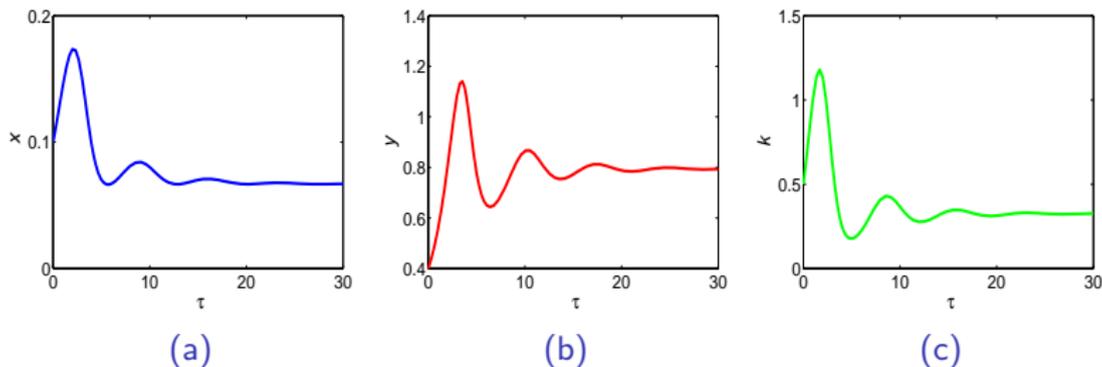


Figure 8 : Time series plots with $\gamma = 2.12$ and $\alpha = 0.3$ for: (a) x ; (b) y ; (c) k . Initial conditions $(x, y, k) = (0.1, 0.4, 0.5)$.

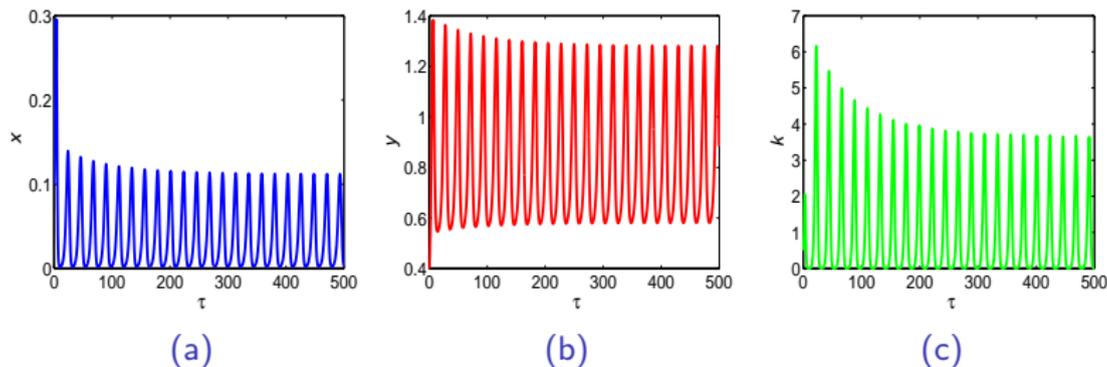


Figure 9 : Time series plots with $\gamma = 2.12$ and $\alpha = 0.003$ for: (a) x ; (b) y ; (c) k . Initial conditions $(x, y, k) = (0.1, 0.4, 0.5)$.

The non-dimensionalised version

$$\begin{aligned}x' &= x \left(1 - \frac{x}{k}\right) - xy, \\y' &= \alpha y \left(1 - \frac{\beta y}{k}\right) + xy, \\k' &= k(\gamma - \delta k - \epsilon x - \phi y).\end{aligned}\tag{2}$$

x : Prey population

y : Predator population

k : Resource

There are 4 equilibria P_i in the (x, y, k) phase space:

Steady states	Characteristics	Stability
$P_1(0, 0, \frac{\gamma}{\delta})$	Extinction of prey and predator	Unstable
$P_2(\frac{\gamma}{\delta+\epsilon}, 0, \frac{\gamma}{\delta+\epsilon})$	Extinction of predator	Unstable
$P_3(0, \frac{\gamma}{\beta\delta+\phi}, \frac{\beta\gamma}{\beta\delta+\phi})$	Extinction of prey	Stable if $\gamma > \beta\delta + \phi$
$P_4(x^*, y^*, k^*)$	Coexistence	Stable if $\gamma < \beta\delta + \phi$

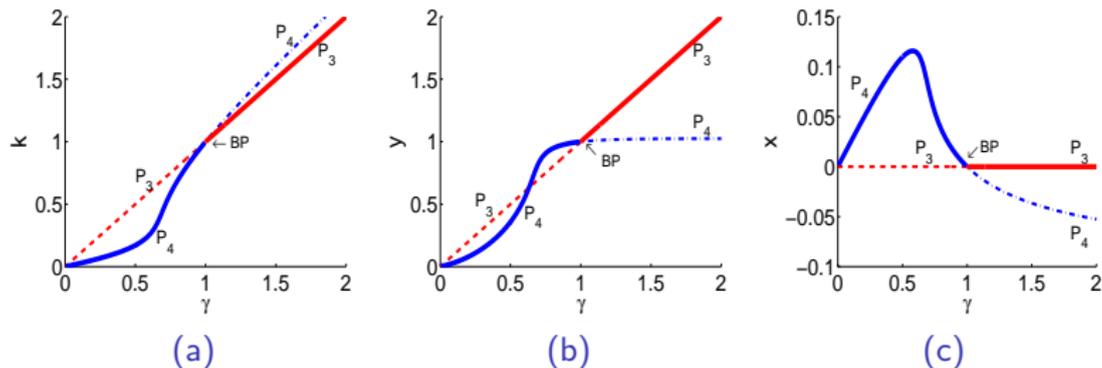


Figure 10 : Steady state diagrams when $\phi = 0, \beta = \delta = 1$ and $\epsilon = 3, \alpha = 0.1$ for (a) k , (b) y , (c) x .

TB : Transcritical Bifurcation;

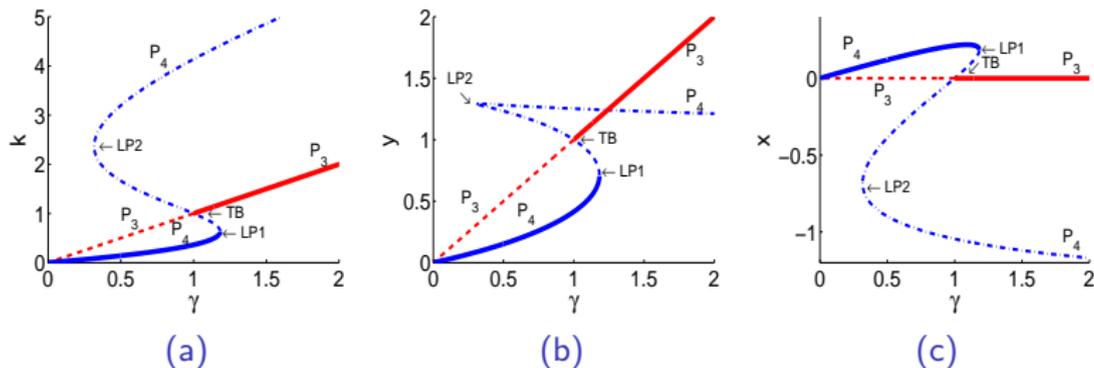


Figure 11 : Steady state diagrams when $\phi = 0, \beta = \delta = 1$ and $\epsilon = 3, \alpha = 1.5$ for (a) k , (b) y , (c) x .

TB : Transcritical Bifurcation; **LP1** : Limit Point 1 ; **LP2** : Limit Point 2.

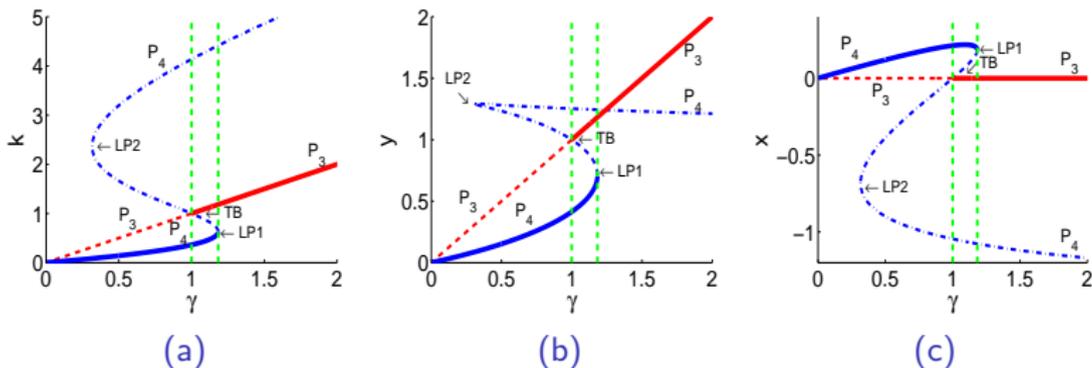


Figure 12 : Steady state diagrams when $\phi = 0, \beta = \delta = 1$ and $\epsilon = 3, \alpha = 1.5$ for (a) k , (b) y , (c) x .

TB : Transcritical Bifurcation; **LP1** : Limit Point 1 ; **LP2** : Limit Point 2.

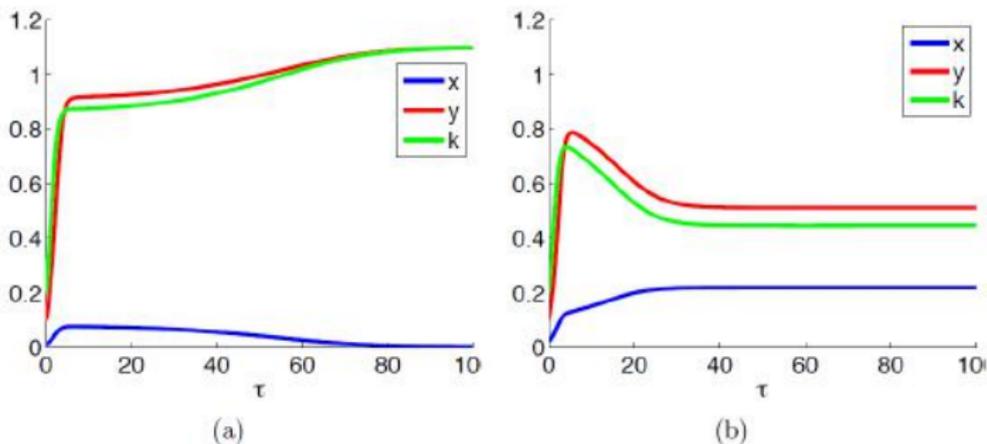


Figure 13 : Time plots of two initial conditions (x_0, y_0, k_0) in the bistability region, $\gamma = 1.1$ for (a) $(0.01, 0.1, 0.2)$ and (b) $(0.02, 0.1, 0.2)$.

By incorporating the harvesting and toxicant effects into a prey-predator fishery model, we can model the non-dimensionalised system of

$$\begin{aligned}x' &= x(1 - x) - \alpha xy - \beta x - \delta x^3, \\y' &= \sigma y(1 - y) + \rho xy - \epsilon y - \mu y^2.\end{aligned}\tag{3}$$

x : Prey fish population

y : Predator fish population

β : Harvesting rate on x

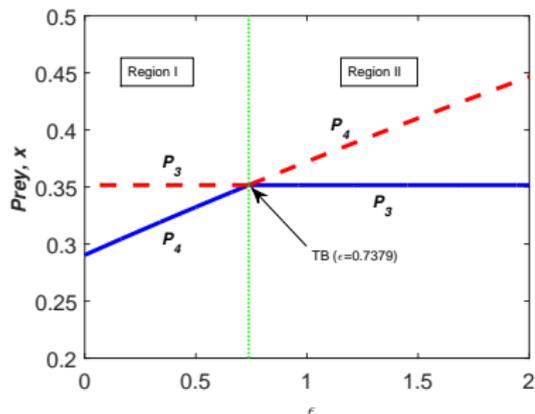
ϵ : Harvesting rate on y

δ : Coefficient of toxicant on x

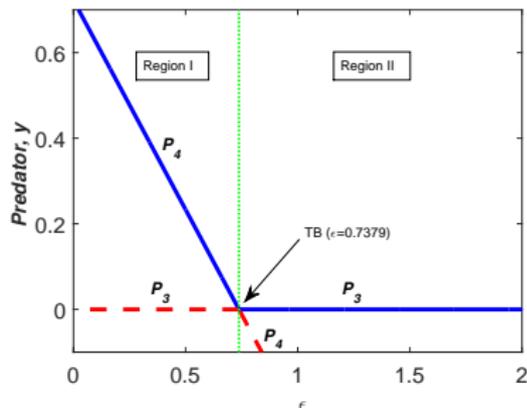
μ : Coefficient of toxicant on y

There are 4 equilibria P_i in the (x, y) phase plane:

Steady states	Characteristics
$P_1(0, 0)$	Extinction of prey and predator
$P_2\left(0, \frac{\sigma - \epsilon}{\rho\sigma}\right)$	Extinction of prey
$P_3\left(\frac{1 - \beta}{\alpha}, 0\right)$	Extinction of predator
$P_4(x^*, y^*)$	Coexistence



(a)



(b)

Figure 14 : Steady state diagrams when $\alpha = 0.15, \beta = 0.5, \delta = 1.2, \sigma = 0.65, \rho = 0.25$ and $\mu = 0.35$ for (a) x , (b) y .

TB : Transcritical Bifurcation;

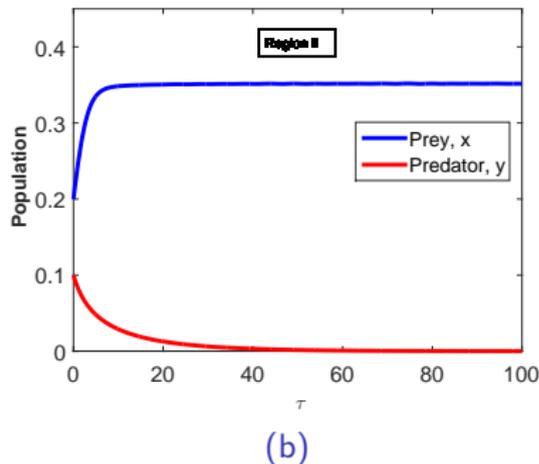
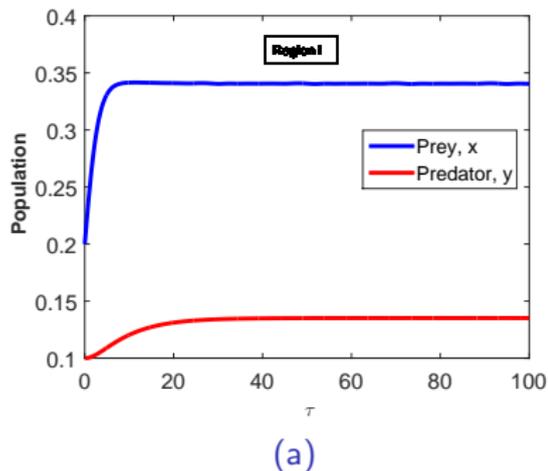


Figure 15 : Time plots with initial conditions of $(x_0, y_0) = (0.2, 0.1)$ at (a) $\epsilon = 0.6$ and (b) $\epsilon = 0.8$.

The non-dimensionalised version

$$\begin{aligned}x' &= x(1 - \alpha x) - xy - \beta x - \sigma x^2 y, \\y' &= -\delta y + xy - \epsilon y - \rho xy^2.\end{aligned}\tag{4}$$

x : Prey fish population

y : Predator fish population

β : Harvesting rate on x

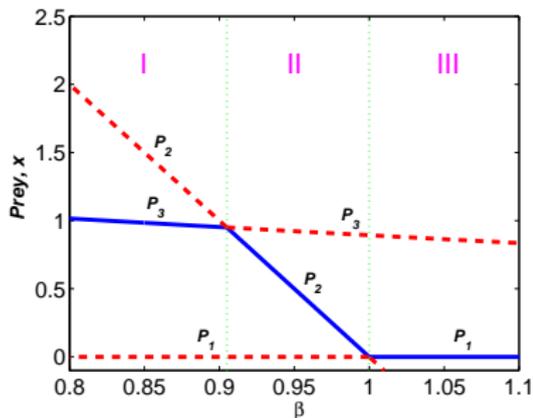
ϵ : Harvesting rate on y

σ : Coefficient of toxicant on x

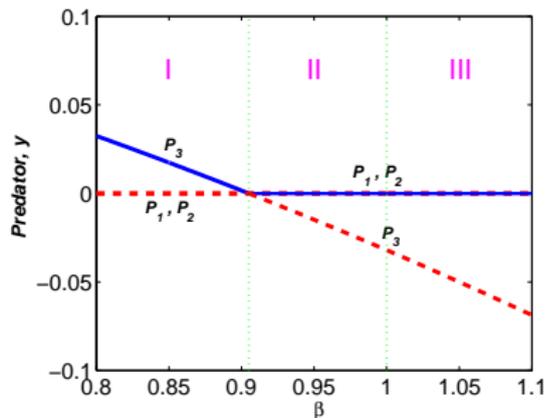
ρ : Coefficient of toxicant on y

There are 3 equilibria P_i in the (x, y) phase plane:

Steady states	Characteristics
$P_1(0, 0)$	Extinction of prey and predator
$P_2\left(\frac{1-\beta}{\alpha}, 0\right)$	Extinction of predator
$P_3(x^*, y^*)$	Coexistence

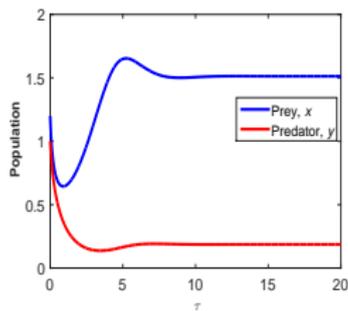


(a)

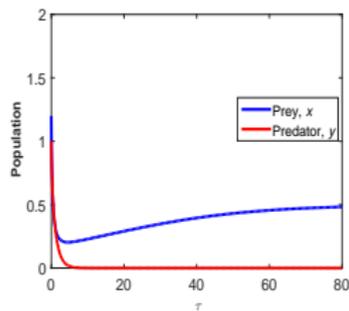


(b)

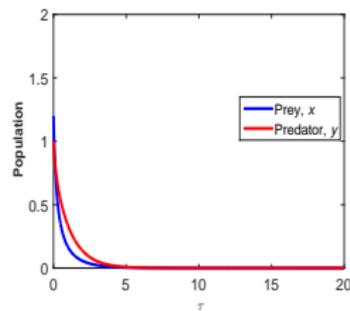
Figure 16 : Steady state diagrams when $\alpha = 0.1$, $\epsilon = 0.85$, $\delta = 0.1$, $\sigma = 2$ and $\rho = 0.2$ for (a) x , (b) y .



(a)



(b)



(c)

Figure 17 : Time plots with initial conditions of $(x_0, y_0) = (1.2, 1.0)$ at (a) $\beta = 0.1$, (b) $\beta = 0.95$, (c) $\beta = 1.8$.

- ▶ Maximum Sustainable Yield (MSY) is widely used in population ecology to determine the largest yield that can be obtained from a certain population without altering dangerously the harvested population.
- ▶ MSY is sometimes unrealistic because it neglects the cost of harvesting.
- ▶ Optimal Harvesting Policy is applied to maximize the profit based on the standard cost benefit criterion while not endangering the harvested species.

- ▶ The economic rent at any time is given by

$$\pi = (pqx - c) E. \quad (5)$$

x : Fish population

p : Unit price of landed fish

q : Catchability coefficient of the fish

c : Cost of harvesting

E : Harvesting Effort

- ▶ The optimal harvesting policy on single fish population is to maximize a continuous time stream of revenues

$$J = \int_0^{\infty} [e^{-\delta t} \cdot \pi] dt. \quad (6)$$

δ : Instantaneous annual rate of discount

π : Economic rent at any time t

- ▶ The continuous revenue function is then optimize by Pontryagin Maximal Principle where the Hamiltonian function is

$$H = [e^{-\delta t} \cdot \pi] + \lambda \left(\frac{dx}{dt} \right). \quad (7)$$

δ : Instantaneous annual rate of discount

λ : Adjoint variable

π : Economic rent at any time t

- ▶ Different from single species model, the economic rent at any time is given by

$$\pi = (p_1 q_1 x + p_2 q_2 y - c) E. \quad (8)$$

x : First fish population y : Second fish population

p_1 : Unit price of landed first fish

q_1 : Catchability coefficient of the first fish

p_2 : Unit price of landed second fish

q_2 : Catchability coefficient of the second fish

c : Cost of harvesting

E : Harvesting Effort

- ▶ Same as single species model, the optimal harvesting policy on multispecies fish population is to maximize a continuous time stream of revenues

$$J = \int_0^{\infty} [e^{-\delta t} \cdot \pi] dt \quad (9)$$

δ : Instantaneous annual rate of discount

π : Economic rent at any time t

However, the function π of multispecies model is different from that of single species.

- ▶ The continuous revenue function is then optimize by Pontryagin Maximal Principle where the Hamiltonian function is

$$H = [e^{-\delta t} \cdot \pi] + \lambda_1 \left(\frac{dx}{dt} \right) + \lambda_2 \left(\frac{dy}{dt} \right) \quad (10)$$

δ : Instantaneous annual rate of discount

λ_1, λ_2 : Adjoint variables

π : Economic rent at any time t

- ▶ Variable carrying capacity plays an crucial role in sustaining both economical and biological growth abd the resource constraint must be taken into account.
- ▶ Harvesting activities on fish population can affect the survival and mortality rate of marine species.
- ▶ Optimal harvesting can be an ideal way to prevent extinction of population in the ecosystem.



THANK YOU

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